Stability of equilibrium states for ferroelectric smectic-C^* liquid crystals in finite and infinite samples

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The aim of this article is to establish some theoretical linear and nonlinear stability results for a dynamic equation that frequently appears in the smectic-C and ferroelectric smectic-C^* liquid crystal literature. We consider finite planar samples confined between bounding plates as well as infinite samples. Many of the results depend on extensions of work for a nonlinear diffusion equation. Critical maximum magnitudes of applied static electric fields are determined, below which stability of a certain constant equilibrium state is ensured.

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I. INTRODUCTION

Many physical and biological processes are modeled well by nonlinear reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u), \quad x \in D, \quad t > 0, \tag{1.1}$$

where D is some domain, possibly infinite. Full nonlinear problems are often analytically intractable and for this reason approximations to more complicated nonlinearities in $f(u)$ are frequently adopted, for example, by cubic nonlinearities. Having made a cubic approximation and the relevant rescalings, one often obtains Nagumo-like equations such as [1]

$$u_t = u_{xx} + u(1-u)(u-a), \quad x \in D, \quad t > 0, \quad 0 < a < 1. \tag{1.2}$$

The model equation to be considered, which arises from smectic liquid crystal theory, is of the form

$$u_t = u_{xx} + a \sin(2u) + b \cos(u), \tag{1.3}$$

where a and b are constants (see, among others, [2], for example). The nonlinear terms in Eq. (1.3) are related to those of the double sine-Gordon equation and a Painlevé analysis of this equation with a any real constant and $b > 0$ has recently been made in [3]. However, it should be pointed out that there is only a first order time derivative in Eq. (1.3) whereas the usual sine-Gordon type of equation generally has a second order time derivative. Flores [4] considered the equation (1.2) on an infinite domain, and states that, by restricting the initial profile, the time dependent nonlinear solution must decay to zero, thereby showing that the zero equilibrium state is nonlinearly stable. The results from [4] are reviewed so that the techniques applied to cubic nonlinearities in Eq. (1.2) can be suitably extended to cover the sinusoidal terms as they occur in Eq. (1.3).

After deriving the relevant dynamic equations for a sample of smectic-C^* liquid crystal in Sec. II we shall discuss results in Sec. III that are used to obtain decay properties for perturbations to Eq. (1.3) that will be used later. The main tool used in proving these decay results is the comparison principle [5–8], which is used to obtain *a priori* bounds on the solution. On obtaining these bounds we can then show that the L_2 norm of the solution decays to zero in time. For completeness we introduce the comparison principle in Sec. III A. In Sec. III B we review in detail Flores' result on an infinite domain in x, before considering in Sec. III C what happens in a finite domain.

The results introduced in Sec. III motivate the style of analysis that will be employed for the liquid crystal problems we investigate in Sec. IV. There we consider a sample of ferroelectric smectic-C^* liquid crystal where a static electric field is applied parallel to the smectic layers. We then apply the methods introduced in Sec. III to obtain information on the stability of the equilibrium state $u = \pi/2$ in Eq. (1.3). Tables I and II in Sec. IV show the stability regimes involving the electric field that are obtained using this method. We obtain a sinusoidal nonlinearity in $f(u)$ which arises in the equation obtained by applying a perturbation, in both space and time, to the equilibrium solution to the dynamic equation derived from the nonlinear continuum theory. Finally, we shall obtain suitable restrictions on the initial data of the perturbation for the linear and nonlinear stability of the $\pi/2$ state of a suitable dynamic equation discussed below, on both infinite and finite domains. Section V contains a discussion of these results and relates the decay properties obtained to the characteristic times τ.

II. GOVERNING EQUATIONS

Liquid crystals are anisotropic fluids consisting of elongated molecules where the long molecular axes locally give rise to a preferred common direction in space, which is usually described by the unit vector **n**, called the director. Ferroelectric smectic-C^* liquid crystals are chiral layered structures possessing a polarization where the director **n** is tilted at an angle θ to the layer normal. We shall assume here that the (temperature dependent) smectic tilt angle θ is some fixed constant, and hence the layers are assumed to be of constant thickness. Having fixed θ, the director is now constrained to rotate around the surface of a fictitious cone. For this reason the smectic tilt θ is often called the cone angle.

Following the description by de Gennes and Prost [9], the orientation of the smectic layers is described by a unit layer normal **a** and a vector **c**, which is the unit orthogonal projection of **n** onto the smectic planes. The direction of the vector

c is described by the angle ϕ which is measured, in the positive sense, relative to the x axis as shown in Fig. 1 (notice that **c** always lies parallel to the smectic planes). Ferroelectric liquid crystals also possess a spontaneous polarization **P** which is assumed locally normal to both **n** and **a**.

The application of an external field to a sample of ferroelectric liquid crystal is known to influence the orientation of the director **n**, and hence of **a** and **c** [9]. We shall now use the continuum theory of Leslie and co-workers [10,11] to obtain the dynamic equation involving $\phi=\phi(z,t)$ for a planar sample of smectic-C^* liquid crystal aligned as in Fig. 1, when an external electric field is applied parallel to the smectic layers in the x direction.

It follows that since the layer normal **a** and the vector **c** are unit and orthogonal to each other they must satisfy the constraints

$$\mathbf{a}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{c}=1, \quad \mathbf{a}\cdot\mathbf{c}=0. \tag{2.1}$$

Since the system that we are considering has constant layer thickness with no dislocations, the layer normal must also satisfy [12]

$$\nabla\times\mathbf{a}=\mathbf{0}. \tag{2.2}$$

For convenience we introduce the vector **b** defined by

$$\mathbf{b}=\mathbf{a}\times\mathbf{c}, \tag{2.3}$$

since the polarization **P** is in the direction of **b**. Further, the director **n** can be expressed in terms of **a** and **c** as

$$\mathbf{n}=\mathbf{a}\cos\theta+\mathbf{c}\sin\theta. \tag{2.4}$$

The dynamic theory of Leslie, Stewart, and Nakagawa [10] involves the construction of a bulk energy integrand involving the **a** and **c** directors. In the **a,c** formulation, the relevant nonchiral contribution to the bulk energy is [11]

$$\begin{aligned}2F_{\mathrm{bulk}}=&A_{21}(\nabla\cdot\mathbf{a})^2+B_1(\mathbf{a}\cdot\nabla\times\mathbf{c})^2+B_2(\nabla\cdot\mathbf{c})^2\\&+B_3(\mathbf{c}\cdot\nabla\times\mathbf{c})^2+(2A_{11}+A_{12}+A_{21}+B_3)(\mathbf{b}\cdot\nabla\times\mathbf{c})^2\\&-(2A_{11}+2A_{21}+B_3)(\nabla\cdot\mathbf{a})(\mathbf{b}\cdot\nabla\times\mathbf{c})\\&-2B_{13}(\mathbf{a}\cdot\nabla\times\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{c})+2(C_1+C_2-B_{13})\\&\times(\nabla\cdot\mathbf{c})(\mathbf{b}\cdot\nabla\times\mathbf{c})-2C_2(\nabla\cdot\mathbf{a})(\nabla\cdot\mathbf{c}),\end{aligned} \tag{2.5}$$

where the A_i, B_i, and C_i are elastic constants. A physical interpretation of these constants and their related deformations is given by Carlsson, Stewart, and Leslie [13]. It is known that the elastic constants A_{12}, A_{21}, B_1, B_2, and B_3 are strictly positive while bounds on the remaining four constants can be derived in terms of these five basic constants. Details can be found in [13] and [14].

Also, since this bulk energy integrand does not take into account the chiral nature of a ferroelectric liquid crystal, we must introduce an additional bulk energy density, which may be expressed, in the **b**,**c** formulation, as [15]

$$F_{\mathrm{chiral}}=\tfrac{1}{2}\Lambda(\mathbf{b}\cdot\nabla\times\mathbf{b}+\mathbf{c}\cdot\nabla\times\mathbf{c})+\tau(\mathbf{c}\cdot\nabla\times\mathbf{c}-\mathbf{b}\cdot\nabla\times\mathbf{b}), \tag{2.6}$$

$$=-\tfrac{1}{2}\Lambda b_i c_{i,j}a_j-\tau b_i a_{i,j}c_j, \tag{2.7}$$

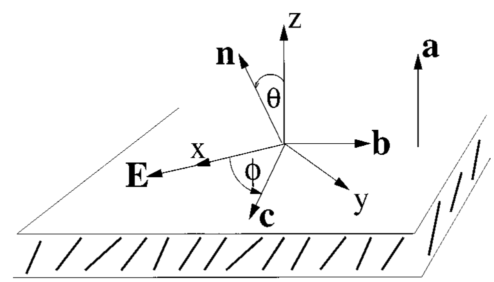

FIG. 1. The geometry of the problem under consideration. The director **n** makes an angle θ with the layer normal **a** with **c** being the unit orthogonal projection of **n** onto the smectic planes, which lie parallel to the xy plane. The z axis coincides with the orientation of the layer normal. The phase angle of the director **c** is denoted by ϕ. The static electric field **E** is applied parallel to the layers in the x direction and **P**, the spontaneous polarization induced by the electric field, is parallel to $\mathbf{b}=\mathbf{a}\times\mathbf{c}$.

where Λ and τ are chiral elastic constants and the latter expressions are in Cartesian component form with repeated indices following the summation convention.

We must also incorporate a term in the energy integrand due to the dielectricity of the smectic-C^* phase. This extra term is [[9], p. 134]

$$F_{\mathrm{elec}}=-\tfrac{1}{2}\varepsilon_a\varepsilon_0(\mathbf{n}\cdot\mathbf{E})^2, \tag{2.8}$$

where ε_0 is the permitivity of free space and ε_a is the dielectric anisotropy of the liquid crystal. A positive dielectric anisotropy indicates that the director prefers to align parallel to the applied field while a negative dielectric anisotropy indicates that the director prefers to orient itself perpendicular to the field. The interaction of the electric field and the spontaneous polarization of the smectic-C^* phase further results in an additional contribution to the energy integrand as

$$F_{\mathrm{pol}}=-\mathbf{P}\cdot\mathbf{E}=-P\mathbf{b}\cdot\mathbf{E}. \tag{2.9}$$

Here we adopt the sign convention, as introduced in [[9], pp. 380, 385], that $P>0$ if the polarization is in the **b** direction.

The total energy integrand can therefore be written as

$$F=F_{\mathrm{bulk}}+F_{\mathrm{chiral}}+F_{\mathrm{elec}}+F_{\mathrm{pol}}. \tag{2.10}$$

Thus the total energy integral over a sample volume V is

$$\mathcal{F}=\int_V F\,dV. \tag{2.11}$$

The relevant dynamic equations in the absence of bulk flow are [10]

$$\mathbf{\Pi}_F^a+\mathbf{g}^a+\lambda\mathbf{a}+\mu\mathbf{c}+\nabla\times\boldsymbol{\beta}=\mathbf{0} \tag{2.12}$$

and

$$\mathbf{\Pi}_F^c+\mathbf{g}^c+\mu\mathbf{a}+\chi\mathbf{c}=\mathbf{0}, \tag{2.13}$$

with, in Cartesian component form,

$$g_i^a=-2\tau_5\dot{c}_i, \quad g_i^c=-2\lambda_5\dot{c}_i, \tag{2.14}$$

where τ_5 and λ_5 are viscosity coefficients ($\lambda_5>0$ is known to be the rotational viscosity coefficient related to the movement of the director **n** around the fictitious cone). The Lagrange multipliers λ, μ, χ, and β arise from the four constraints in Eqs. (2.1) and (2.2). The vectors $\mathbf{\Pi}_F^a$ and $\mathbf{\Pi}_F^c$ in Eqs. (2.12) and (2.13) are defined by

$$\{\mathbf{\Pi}_F^a\}_i=\left\{\frac{\partial F}{\partial a_{i,j}}\right\}_{,j}-\frac{\partial F}{\partial a_i} \qquad (2.15)$$

and

$$\{\mathbf{\Pi}_F^c\}_i=\left\{\frac{\partial F}{\partial c_{i,j}}\right\}_{,j}-\frac{\partial F}{\partial c_i}. \qquad (2.16)$$

Introducing the ansatz

$$\mathbf{a}=(0,0,1), \qquad (2.17)$$

$$\mathbf{c}=(\cos\phi, \sin\phi,0), \qquad (2.18)$$

and

$$\mathbf{E}=(E,0,0), \qquad (2.19)$$

we obtain, eliminating the Lagrange multipliers in Eqs. (2.12) and (2.13) in a similar fashion to that contained in the Appendix in [16], the governing equation for the phase angle ϕ,

$$2\lambda_5\frac{\partial\phi}{\partial t}-B_3\frac{\partial^2\phi}{\partial z^2}+E^2\varepsilon_a\varepsilon_0\sin^2(\theta)\sin\phi\cos\phi+PE\cos\phi =0, \qquad (2.20)$$

which describes the realignment of the director. Thus, on rescaling Eq. (2.20), we obtain the dynamic equation

$$\phi_T=\phi_{ZZ}-A\sin\phi\cos\phi-B\cos\phi, \qquad (2.21)$$

where we have introduced the constants

$$A=E^2\varepsilon_a\varepsilon_0\sin^2(\theta), \quad B=PE, \qquad (2.22)$$

and the rescaled variables

$$T=\frac{1}{2\lambda_5}t, \qquad (2.23)$$

and

$$Z=\frac{1}{\sqrt{B_3}}z. \qquad (2.24)$$

Equation (2.21) is similar to the form of the governing equation used by Maclennan, Clark, and Handschy [17] and is known to arise in the modeling of surface stabilized ferroelectric liquid crystal devices. However, in [17], alignment is described in an equivalent way in terms of **a** and **P** rather than **a** and **c**. The direction of the polarization **P** is then described by a phase angle which is measured in the same sense as ϕ introduced above but with a phase shift of $\pi/2$ (see Fig. 1). We choose to work in terms of ϕ defined in Fig. 1 since it sets the problem in a slightly more general setting and therefore allows comparisons to be drawn with other work in Refs. [3,16–21] (see especially the Appendix of [3]).

At this point it is instructive to highlight the role of the electric potential in relation to the critical field magnitude which will be calculated and discussed below, especially E_c given by Eq. (4.13). Using the above definitions for the vectors in Eqs. (2.3), (2.4), (2.17), (2.18), and (2.19), we can consider a sequence of qualitative plots for the combined electric potential $u(\phi)$ for Eqs. (2.8) and (2.9) given by

$$u(\phi)=F_{\mathrm{elec}}+F_{\mathrm{pol}}=-\tfrac{1}{2}\varepsilon_a\varepsilon_0E^2\sin^2(\theta)\cos^2\phi+PE\sin\phi, \qquad (2.25)$$

keeping ε_a, ε_0, θ, and P fixed. For convenience, introduce the constant

$$E^*=\frac{P}{|\varepsilon_a|\varepsilon_0\sin^2(\theta)}. \qquad (2.26)$$

Elementary calculations reveal that there are either two or three real turning points for $u(\phi)$, namely,

$$\phi=\begin{cases}\dfrac{\pi}{2},\dfrac{3\pi}{2} & \text{whenever } |E|\leqslant E^* \\[2ex] \dfrac{\pi}{2},\dfrac{3\pi}{2},\arcsin\left(\dfrac{-P}{E\varepsilon_a\varepsilon_0\sin^2(\theta)}\right) & \text{whenever } |E|>E^*.\end{cases} \qquad (2.27)$$

The nature of $u(\phi)$ is demonstrated in Fig. 2. There is only one local minimum at $\phi=3\pi/2$ for $E\leqslant E^*$ while if $E>E^*$ there are two local minima at $\pi/2$ and $\arcsin[-P/E\varepsilon_a\varepsilon_0\sin^2(\theta)]$, both giving equal values for the potential $u(\phi)$: thus the system is expected to change from having one possible stable state to having two possible states of equal potential, that is, for $E>E^*$ the system can exhibit bistability. The relationship and consequences of E^* in Eq. (2.26) upon the stability of solutions to dynamic equations such as Eq. (2.20) are discussed in detail in Sec. IV below. It should also be noted that Eq. (2.20) may also be obtained by considering a balance of elastic and electric torques, in a similar way to the analysis carried out by Schiller, Pelzl, and Demus [20].

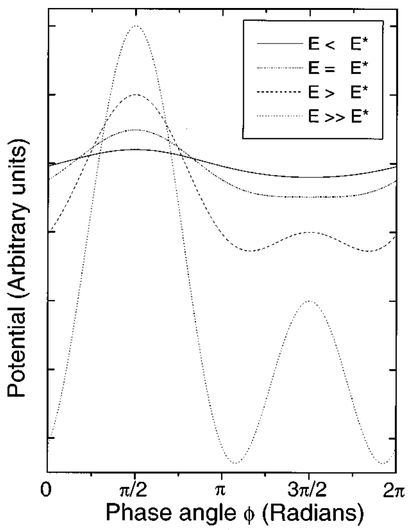

FIG. 2. Qualitative plots of the total electric potential $u(\phi)$ in Eq. (2.25) as a function of the phase angle ϕ. When $E \leqslant E^*$ the potential exhibits one local minimum at $\phi = 3\pi/2$. However, if $E > E^*$ given by Eq. (2.26) then the minimum at $\phi = 3\pi/2$ becomes a maximum and two new minima appear at $\pi/2$ and $\arcsin[-P/E\varepsilon_a\varepsilon_0 \sin^2(\theta)]$, which possess equal potential values, showing that the system can exhibit bistability.

III. COMPARISON PRINCIPLES AND PRELIMINARY RESULTS ON CUBIC NONLINEARITIES

In this section we state and review key results that are exploited in the subsequent sections.

A. Comparison principles

The main tools we shall use involve comparison principles for partial differential equations. Hence for clarity of exposition and convenience we briefly summarize in this section the comparison principle for both finite and infinite domains. We begin by considering the differential equation

$$v_t = v_{xx} + g(v,x,t), \quad x \in \Omega, \quad t > 0, \tag{3.1}$$

where Ω can be either the whole, or a strict subset, of $\mathbb{R}$, and g is assumed to be continuously differentiable. We summarize the basic results on the comparison principle (see, for example, [[5], pp. 54–56]).

A *supersolution* is a function $\overline{v}$: $\Omega \times [0,T] \to B$, for some bounded subset B of $\mathbb{R}$, such that

$$\overline{v}_t \geqslant \overline{v}_{xx} + g(\overline{v},x,t). \tag{3.2}$$

Similarly, a *subsolution* $\underline{v}$ is a function $\underline{v}$: $\Omega \times [0,T] \to B$, and

$$\underline{v}_t \leqslant \underline{v}_{xx} + g(\underline{v},x,t). \tag{3.3}$$

Now suppose that initially we have

$$\overline{v}(x,0) \geqslant \underline{v}(x,0). \tag{3.4}$$

If Ω is an infinite domain, for example, $\Omega \equiv \mathbb{R}$, then we have that the super- and subsolutions satisfy

$$\overline{v}(x,t) \geqslant \underline{v}(x,t), \quad x \in \Omega, \quad t \in [0,T]. \tag{3.5}$$

If, however, Ω is a bounded subset of $\mathbb{R}$, then we must also impose an extra condition which takes into consideration the behavior of the solution at the boundary. Hence, we must also determine if there exist constants α,β $(\alpha^2+\beta^2 \neq 0)$ such that

$$\alpha\overline{v} - \beta\overline{v}_{xx} \geqslant \alpha\underline{v} - \beta\underline{v}_{xx}, \quad x \in \partial\Omega, \quad t > 0. \tag{3.6}$$

Thus, on a finite domain, if $\overline{v}$ and $\underline{v}$ are super-and subsolutions, respectively, satisfying condition (3.4) and the condition on the boundary (3.6), then

$$\overline{v}(x,t) \geqslant \underline{v}(x,t), \quad x \in \Omega, \quad t \in [0,T]. \tag{3.7}$$

B. The cubic nonlinearity on an infinite domain

We shall now, as was discussed in [4], investigate the stability of the zero equilibrium solution to a dynamic equation that has a cubic nonlinearity. We begin by considering the simpler cubic nonlinearity case, before moving on to consider a more complicated sinusoidal nonlinearity in Sec. IV, in order to obtain explicit decay properties that we shall use in Sec. IV.

The stability analysis that we consider here involves introducing a perturbation $u_0(x)$ at time $t=0$ and examining the ensuing time dependent behavior. We begin, as in [4], by considering the Nagumo equation on an infinite domain,

$$u_t = u_{xx} + u(1-u)(u-a), \quad x \in D, \quad t > 0, \quad 0 < a < 1, \tag{3.8}$$

$$u(x,0) = u_0(x) \in H^1, \tag{3.9}$$

where, for our purposes, a is a constant, $u_0(x)$ is a nonnegative initial profile, and H^1 is the usual Hilbert space of functions that, with their first derivatives, belong to the space of real square integrable functions $L_2(\mathbb{R})$. By the Sobolev embedding theorem, this also implies that $u_0 \in C_B(\mathbb{R})$, the space of continuous bounded functions (see, for example, [[22], pp. 95–97]).

Local existence is guaranteed by standard Lipschitz arguments (see, for example, [[5], p. 46]); therefore there exists $u(x,t)$ on $\mathbb{R} \times [0,T]$ for some $T > 0$. The time dependent solution $u(x,t)$ is also therefore restricted to lie in the function space H^1 for $t \in [0,T]$.

Let $u_c(x,t)$ be such a solution to Eq. (3.8) satisfying

$$u_c(x,0) = cu_0(x), \tag{3.10}$$

where c is some positive constant. It is possible to choose c small enough so that

$$cu_0(x) \leqslant a_0 < a, \quad x \in \mathbb{R}, \tag{3.11}$$

where a_0 is any positive constant strictly less than a. We now wish to apply a comparison principle, to obtain lower and upper bounds on the solution $u_c(x,t)$.

Let

$$\overline{u}(x,t) = a_0 \tag{3.12}$$

and

$$\underline{u}(x,t) = u_c(x,t). \tag{3.13}$$

Upon substituting Eqs. (3.12) and (3.13) into Eq. (3.8), we obtain the inequalities

$$\overline{u}_t \geqslant \overline{u}_{xx} + \overline{u}(1-\overline{u})(\overline{u}-a), \quad x \in \mathbb{R}, \quad t \in [0,T], \tag{3.14}$$

and

$$\underline{u}_t \leqslant \underline{u}_{xx} + \underline{u}(1-\underline{u})(\underline{u}-a), \quad x \in \mathbb{R}, \quad t \in [0,T]. \tag{3.15}$$

Thus $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are super- and subsolutions, respectively. Note that equality actually holds in Eq. (3.15). Since we are dealing with an infinite domain, all that remains is to show that the super- and subsolutions satisfy the inequalities in Eq. (3.11) for $x \in \mathbb{R}$ at $t=0$, so that the corresponding inequality (3.4) holds. Consideration of these initial states reveals that, by the judicious choice of the constant c in Eq. (3.7),

$$\overline{u}(x,0) = a_0 \geqslant c u_0(x) = \underline{u}(x,0). \tag{3.16}$$

Hence, it follows by the comparison principle in Sec. III A that

$$a_0 = \overline{u}(x,t) \geqslant \underline{u}(x,t) = u_c(x,t), \quad x \in \mathbb{R}, \quad t \in [0,T]. \tag{3.17}$$

However, since (3.14) and (3.15) hold for any $T>0$, Eq. (3.17) can be extended to hold globally [[5], p. 55], that is,

$$u_c(x,t) \leqslant a_0, \quad x \in \mathbb{R}, \quad t>0. \tag{3.18}$$

If we now choose

$$\overline{u}(x,t) = u_c(x,t) \tag{3.19}$$

and

$$\underline{u}(x,t) = 0, \tag{3.20}$$

it is possible to bound the solution $u_c(x,t)$ below, for all time, by zero by a similar application of the comparison principle. Thus we have obtained the bounds

$$0 \leqslant u_c(x,t) \leqslant a_0 < a < 1, \quad x \in \mathbb{R}, \quad t \in [0,\infty), \tag{3.21}$$

indicating that our solution $u_c(x,t)$ must be non-negative.

More qualitative information on $u_c(x,t)$ can be obtained by employing the techniques of Flores [4]. It follows that, since our solution u_c is bounded above by a_0,

$$(1-u_c)(u_c-a) \leqslant (1-a_0)(a_0-a), \quad x \in \mathbb{R}, \quad t>0, \tag{3.22}$$

and thus the nonlinear term in (3.8) satisfies

$$u_c(1-u_c)(u_c-a) \leqslant -k u_c, \tag{3.23}$$

where k is the positive constant

$$k = (1-a_0)(a-a_0). \tag{3.24}$$

It follows from (3.23) that

$$\frac{\partial u_c}{\partial t} = \frac{\partial^2 u_c}{\partial x^2} - k u_c \geqslant \frac{\partial^2 u_c}{\partial x^2} + u_c(1-u_c)(u_c-a), \quad x \in \mathbb{R}, \quad t>0, \tag{3.25}$$

and hence the solution to

$$U_t = U_{xx} - kU, \quad x \in \mathbb{R}, \quad t>0,$$
$$U(x,0) = C u_0(x), \tag{3.26}$$

where $C \geqslant c$, must be a supersolution, by the definition in Sec. III A. Equation (3.26) can be reduced to the canonical heat equation by making the substitution

$$\eta(x,t) = U(x,t) e^{kt}, \tag{3.27}$$

and so

$$\eta_t = \eta_{xx},$$
$$\eta(x,0) = U(x,0) = C u_0(x). \tag{3.28}$$

Equation (3.28) has the well known solution

$$\eta(x,t) = C \int_{-\infty}^{\infty} K(x-y,t) u_0(y) dy, \tag{3.29}$$

where

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-x^2}{4t} \right) \tag{3.30}$$

is the usual fundamental solution to the heat equation [23]. Thus we have

$$U(x,t) = \frac{C e^{-kt}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4t} \right) u_0(y) dy, \tag{3.31}$$

which is a supersolution. Hence, choosing $u_c(x,t)$ as a subsolution, we can bound our solution above by $U(x,t)$, provided these super- and subsolutions have the correct behavior at $t=0$.

On the boundary $x \in \mathbb{R}$, $t=0$, we have that

$$U(x,0) = C u_0(x) \geqslant c u_0(x) = u_c(x,0), \tag{3.32}$$

and thus we can apply the comparison principle to obtain

$$U(x,t) \geqslant u_c(x,t), \quad x \in \mathbb{R}, \quad t>0,$$

that is, we can now bound our non-negative solution above by a function that exhibits an exploitable time dependence, namely,

$$u_c(x,t) \leqslant \frac{C e^{-kt}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4t} \right) u_0(y) dy. \tag{3.33}$$

Having obtained this upper bound on the solution $u_c(x,t)$, we now consider the L_2 norm of $u_c(x,t)$. Since we have seen from Eq. (3.21) that $u_c(x,t)$ is non-negative, it then follows that, if the L_2 norm of the solution decreases in time, the zero state is nonlinearly stable to initial perturbations satisfying Eq. (3.11). The L_2 norm of $u_c(x,t)$ is defined to be

$$\|u_c(x,t)\|_{L_2}^2 = \int_{-\infty}^{\infty} u_c^2(x,t)dx, \quad t>0, \tag{3.34}$$

and thus, on using (3.33), we obtain

$$\|u_c(t)\|_{L_2}^2 \leqslant e^{-kt}\int_{-\infty}^{\infty} u_c(x,t)\psi(x,t)dx, \tag{3.35}$$

where

$$\psi(x,t) = C\int_{-\infty}^{\infty} K(x-y,t)u_0(y)dy. \tag{3.36}$$

Hence (3.35) implies that

$$\|u_c(t)\|_{L_2}^4 \leqslant e^{-2kt}\left(\int_{-\infty}^{\infty} u_c(x,t)\psi(x,t)dx\right)^2. \tag{3.37}$$

We now state a standard result from [[24], p. 528] that we shall require. If $f\in L_1$ and $g\in L_2$ then we have that the convolution

$$h = \int_{-\infty}^{\infty} f(x-y)g(y)dy \in L_2 \quad \text{and } \|h\|_{L_2} \leqslant \|g\|_{L_2}\|f\|_{L_1}. \tag{3.38}$$

Now consider the convolution

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\exp\left(\frac{-(x-y)^2}{4t}\right)Cu_0(y)dy. \tag{3.39}$$

By assumption we have that $Cu_0(x)$ is square integrable on $\mathbb{R}$ and it is clear that $K(x,t)$ in Eq. (3.30) is integrable on $\mathbb{R}$ for $t>0$. Thus, applying the result in Eq. (3.38), we find that the convolution in Eq. (3.39) must be square integrable.

Knowing that the integral in (3.39) is an L_2 function, which shows that $u_c(x,t)\psi(x,t)$ is an L_1 function, now enables us to apply the well known Cauchy-Schwartz inequality to the integral within the large parentheses in Eq. (3.37), showing that

$$\left(\int_{-\infty}^{\infty}[u_c(x,t)\psi(x,t)]dx\right)^2 \leqslant \int_{-\infty}^{\infty} u_c^2(x,t)dx \int_{-\infty}^{\infty}\psi^2(x,t)dx$$
$$= \|u_c\|_{L_2}^2 \int_{-\infty}^{\infty}\psi^2(x,t)dx. \tag{3.40}$$

Since we have that $K(x,t)$ is integrable and $u_0(x)$ is square integrable, we have, on applying the result in Eq. (3.38) to $\psi(x,t)$, that

$$\int_{-\infty}^{\infty}\psi^2(x,t)dx \leqslant C^2\int_{-\infty}^{\infty} u_0^2(x)dx\int_{-\infty}^{\infty} K(x,t)dx, \tag{3.41}$$

which, on noting (see, for example, [[23], p. 34]) that for $t>0$ the fundamental solution satisfies

$$\int_{-\infty}^{\infty} K(x,t)dx = 1, \tag{3.42}$$

reveals that

$$\int_{-\infty}^{\infty}\psi^2(x,t)dx \leqslant C^2\|u_0\|_{L_2}^2. \tag{3.43}$$

Thus, combining inequalities (3.37), (3.40), and (3.43) we have that

$$\|u_c(x,t)\|_{L_2}^4 \leqslant C^2\|u_c(x,t)\|_{L_2}^2\|u_0(x)\|_{L_2}^2 e^{-2kt},$$

which implies

$$\|u_c(x,t)\|_{L_2} \leqslant Me^{-kt}, \tag{3.44}$$

where

$$M = C\|u_0\|_{L_2} < \infty. \tag{3.45}$$

Thus, we see from inequalities (3.44) and (3.45) that a solution corresponding to small initial data collapses, that is, the zero state solution to Eq. (3.8), is nonlinearly stable to any positive initial perturbations in H^1 satisfying (3.11).

C. The cubic nonlinearity on a finite interval

Having considered data collapse for the Nagumo equation (3.8) on an infinite domain, we now consider the possibility of data collapse in a finite domain *D*. In particular, we consider a finite closed interval *D* in *x*: this corresponds to the usual ‘‘bookshelf geometry.’’

A general stability result for equations of the form of Eq. (1.1) can be found in [[25], p. 158]. In [25] the stability result depends upon $f(u)$ satisfying certain given properties. If these restrictions are satisfied, this result guarantees the existence of a finite constant such that if the L_2 norm of the initial profile of *u* is less than or equal to this constant, the solution decays exponentially in the more restrictive space $H_0^1\cap C_0$. Moreover, a bound upon the decay time, which involves the first eigenvalue of the Laplacian, is obtained. This result can be used to show that, providing certain restrictions hold, the solution will always decay to zero. However, no information can be found on the maximum magnitude of the initial profile for which stability will hold. Also, there is no relation between the relaxation time and the magnitude of the initial profile. For this reason we choose to analyze the finite case in a similar way to that considered in the infinite case in order to obtain more detailed behavior of the decay properties. As in the infinite case, we consider the solution $u_c(x,t)$ that satisfies Eq. (3.10). Since we are interested here in the stability of the zero state, we also impose the extra restriction that $u_0(x)$, and thus $u_c(x,t)$, vanishes on the boundary.

Analogous to the infinite case we now choose *c* small enough so that

$$cu_0(x) \leqslant a_0 < a, \quad x\in D. \tag{3.46}$$

We now wish to use the comparison principle on a finite domain to obtain upper and lower bounds on the solution. It follows from the comparison principle that, if the sub- and supersolutions used in the infinite domain case are chosen here to obtain upper and lower bounds on the solution

$u_c(x,t)$, then these sub- and supersolutions must also satisfy the required boundary conditions (3.6).

First, we consider

$$\overline{u}(x,t)=a_0 \tag{3.47}$$

and

$$\underline{u}(x,t)=u_c(x,t), \tag{3.48}$$

with, as before, u_c being any solution to (3.8) satisfying $u_c(x,0)=cu_0(x)$ on D. On the boundary of D we have that

$$\overline{u}=a_0\geqslant 0=\underline{u}=u_c(x,t), \quad x\in\partial D, \quad t>0. \tag{3.49}$$

Hence it follows that, since we have already seen in the infinite case that these choices of $\overline{u}$ and $\underline{u}$ are super- and subsolutions, respectively, satisfying the requirements at $t=0$ for the comparison principle, we can now apply the finite version of the comparison principle to obtain [incorporating the additional conditions on ∂D in Eq. (3.6)]

$$u_c(x,t)\leqslant a_0, \quad x\in D, \quad t>0. \tag{3.50}$$

We are able to extend this bound for all time since $\overline{u}$ and $\underline{u}$ are super- and subsolutions for all $t>0$. We now consider the choice of

$$\overline{u}=u_c(x,t) \tag{3.51}$$

and

$$\underline{u}=0. \tag{3.52}$$

On ∂D we have that

$$\overline{u}=0\geqslant 0=\underline{u}. \tag{3.53}$$

Thus we have by the finite comparison principle

$$0\leqslant u_c(x,t), \quad x\in D, \quad t>0. \tag{3.54}$$

Combining (3.50) and (3.54), yields

$$0\leqslant u_c(x,t)\leqslant a_0, \quad x\in D, \quad t>0. \tag{3.55}$$

Having bounded our solution above by a_0, it again follows that we can bound the nonlinearity in Eq. (3.8). Indeed, the bound given in (3.23) holds and this leads us to conclude that the solution to

$$U_t=U_{xx}-kU, \quad x\in D, \quad t>0 \tag{3.56}$$

is a supersolution. As in the infinite case, we again reduce Eq. (3.56) to the heat equation in terms of η by using the substitution (3.27). However, the solution to the heat equation on a finite domain is now given in terms of an infinite series (see, for example, [[23], p. 43]). On making the substitution (3.27), Eq. (3.56) is reduced to the canonical heat equation (3.28), where without loss of generality we assume that $x\in[0,d]$, where d is the depth of the given sample. Introducing the rescaled variables

$$X=\frac{x}{d} \tag{3.57}$$

and

$$T=\frac{t}{d^2}, \tag{3.58}$$

we have that

$$\eta=\sum_{n=1}^{\infty} A_n \exp[-(n\pi)^2T]\sin(n\pi X), \tag{3.59}$$

where

$$A_n=2\int_0^1 Cu_0(X)\sin(n\pi X)dX, \quad n=1,2,3,\ldots. \tag{3.60}$$

It therefore follows that, on combining Eqs. (3.27) and (3.59), we obtain

$$u_c(x,t)\leqslant \exp(-kt)\sum_{n=1}^{\infty} A_n \exp\left(-\frac{(n\pi)^2t}{d^2}\right)\sin\left(\frac{n\pi x}{d}\right). \tag{3.61}$$

Since $u_0(x)$ and $\sin(n\pi x/d)$ are bounded, we have that each of the A_n are also bounded; in fact

$$-2Ca_0\leqslant A_n\leqslant 2Ca_0, \quad n=1,2,3,\ldots. \tag{3.62}$$

At $t=0$ the Fourier series satisfies

$$u_c(x,0)\leqslant\sum_{n=1}^{\infty} A_n \sin\left(\frac{(n\pi x)}{d}\right)=cu_0(x)\leqslant ca_0, \tag{3.63}$$

while for $t>0$, upon using the upper bound on the A_n given in (3.62),

$$u_c(x,t)\leqslant 2a_0C\exp\{-kt\}\sum_{n=1}^{\infty} \exp\left\{-\frac{\{n\pi\}^2t}{d^2}\right\}. \tag{3.64}$$

By applying the usual ratio test we have that the infinite sum on the right hand side of (3.64) is convergent to a finite limit, $L(t)$ say. Thus, for $t>t_1$ where t_1 is some fixed positive constant, we have that

$$u_c(x,t)\leqslant M_1\exp(-kt), \quad t>t_1, \tag{3.65}$$

where M_1 is a uniform bound on the above sum that holds for $t\geqslant t_1>0$, namely,

$$M_1=2a_0CL(t_1). \tag{3.66}$$

Having bounded our function uniformly for $t\geqslant t_1$ by Eq. (3.66) it follows, from continuity in t, that $u_c(x,t)$ is bounded on $[0,t_1]$, by some constant M_2. Let

$$M=\max(M_1,M_2). \tag{3.67}$$

Then, taking the L_2 norm of $u_c(x,t)$, we have

$$\|u_c(x,t)\|_{L_2}^2\leqslant\int_0^d M^2\exp(-2kt)dx=(Md)^2\exp(-2kt), \tag{3.68}$$

which implies that

$$\|u_c(x,t)\|_{L_2} \leqslant \bar{M} \exp(-kt), \quad t \geqslant 0, \tag{3.69}$$

where $\bar{M} = Md$. Hence, since from (3.55) $u_c(x,t)$ is non-negative, we have that on a finite interval in x, solutions with small enough initial profiles collapse to zero. Therefore, as in the infinite domain case, the zero equilibrium solution is nonlinearly stable to initial perturbations satisfying (3.46). We notice here that the decay rate found from (3.69) is not related to the sample depth.

IV. STABILITY FOR FERROELECTRIC SMECTIC-C^* LIQUID CRYSTALS

Having reviewed and developed the analytic techniques in Secs. III B and III C to deduce whether or not solutions to diffusion equations with cubic nonlinearities collapse to zero and, if they do, what their decay rate is like, we now consider applying these methods to a more complicated case when the diffusion equation has a sinusoidal nonlinearity. The sinusoidal nonlinearity to be considered is obtained from applying a perturbation analysis to one of the constant equilibrium states of the dynamic equation derived in Sec. II for ferroelectric smectic-C^* liquid crystals. This sinusoidal nonlinearity must be considered separately from the cubic cases of Secs. III B and III C as it is not possible to reduce Eq. (2.21) to the form of Eq. (3.8) via a substitution.

In this particular case where the static field is only being considered applied parallel to the smectic layers, the techniques that we employ cannot be applied to the nonconstant equilibrium states. In such cases, the nonlinearity obtained in the perturbation equation, which depends not only upon the perturbation but also upon the equilibrium state, cannot satisfy the bounds that are required to enable the application of the comparison principles of Sec. III A. We therefore consider only the stability of the constant equilibrium solution $\pi/2$ to Eq. (2.20).

As derived in Eqs. (2.21)–(2.24), the governing equation for a sample of smectic-C^* liquid crystal with an electric field applied parallel to the layers is given by

$$\phi_T = \phi_{ZZ} - A \sin\phi \cos\phi - B \cos\phi, \quad Z \in D, \quad T > 0,$$

$$\phi(Z,0) = \phi_0(Z), \tag{4.1}$$

where D can be either finite or infinite.

Before applying the methods introduced in Secs. III B and III C to obtain stability results for the above problem we must first of all define what is meant by an equilibrium state $\hat{\phi}(Z)$ being stable. We first introduce a perturbation $w(Z,T)$, in both space and time, satisfying [5,7]

$$\phi(Z,T) = \hat{\phi}(Z) + w(Z,T) \tag{4.2}$$

(see Fig. 3). The equilibrium state $\hat{\phi}(z)$ is then defined to be stable if

$$\|w(Z,T)\|_{L_2} \to 0 \quad \text{as} \quad T \to \infty. \tag{4.3}$$

Thus on substituting Eq. (4.2) with $\hat{\phi} = \pi/2$ into Eq. (4.1) we obtain the nonlinear dynamic equation for w, namely,

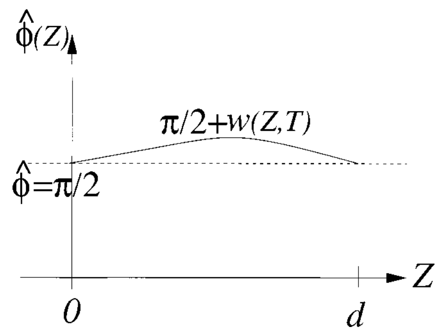

FIG. 3. Schematic of a possible perturbation on a finite domain $[0,d]$.

$$w_T = w_{ZZ} + A\cos(w)\sin(w) + B\sin(w), \tag{4.4}$$

$$w(Z,0) = w_0(Z) \in H^1, \tag{4.5}$$

which governs the growth of the perturbation $w(Z,T)$. We shall consider two types of stability here. The first type of stability to be examined is linear stability. For linear stability we assume that the perturbation $w(Z,T)$ is small and it is therefore possible to linearize the nonlinearity in Eq. (4.4). We then consider conditions for the solution of this linearized equation to decay. Secondly, we will consider the stability of the solution to the fully nonlinear problem (4.4) and obtain restrictions on the strength of the applied static field for stability to hold.

Since the cubic nonlinearity

$$q(u) = u(1-u)(u-a) \tag{4.6}$$

in Secs. III B and III C becomes negatively unbounded we are required to assume, in order to obtain a negative lower bound on $f(u)$ as in [4], that the initial profile $u_0(x)$ is non-negative. In the present problem, the nonlinearity on the right hand side of Eq. (4.4) involving the perturbation $w(Z,T)$ remains bounded for all values of $w(Z,T)$. It is not possible, however, due to the behavior of the nonlinearity around $w=0$ on the right hand side of Eq. (4.4), to obtain a uniform negative lower bound on the nonlinearity for both $w<0$ and $w>0$. We cannot therefore consider perturbations w that change sign. We shall discuss below the case when w is non-negative with the case for $w<0$ being similar. This work is analogous to a first eigenmode approximation.

A. Linear stability

Assuming that $w(Z,T)$ is small and linearizing the nonlinearity in Eq. (4.4), we obtain the linearized perturbation problem

$$w_T = w_{ZZ} + (A+B)w, \tag{4.7}$$

with A and B given by Eq. (2.22). By making the substitution

$$\eta(Z,T) = \exp(-[A+B]T)w(Z,T), \tag{4.8}$$

it is now possible to reduce Eq. (4.7) to the heat equation

$$\eta_T = \eta_{ZZ},$$

$$\eta(Z,0) = w_0(Z). \tag{4.9}$$

TABLE I. Ranges of stability for $E>0$ in the infinite case where E_c is given by Eq. (4.13), ε_a is the dielectric anisotropy of the liquid crystal, and P is the spontaneous polarization.

	P	
ε_a	$+$	$-$
$+$	method fails	$0<E<E_c$
$-$	$E>E_c$	$E>0$

Case (i): Infinite domain

We first consider the infinite domain problem. Equation (4.9) on the infinite domain has a solution similar to (3.31). Thus

$$w(Z,T)=\exp([A+B]T)\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi T}}\times\exp\left(\frac{-(Z-y)^2}{4T}\right)w_0(y)dy. \quad (4.10)$$

Applying the arguments introduced in the infinite cubic case in Sec. III B it is now possible to bound our solution [cf. (3.44)], to obtain

$$\|w\|_{L_2}\leqslant M\exp([A+B]T), \quad (4.11)$$

where M is a finite constant. We see from Eq. (4.11) that the growth of the perturbation w and hence the stability of the solution $\phi(Z,T)$ are dependent upon the sign of

$$A+B=E(P+E\varepsilon_a\varepsilon_0\sin^2\theta). \quad (4.12)$$

Therefore it follows that for the $\phi=\pi/2$ state to be stable $A+B$ must be negative. The positivity or negativity of the right hand side of Eq. (4.12), and therefore the stability of the $\pi/2$ equilibrium state, are dependent upon the positivity of E, P, and ε_a. A maximum critical magnitude of the applied static field E_c can be calculated by solving the quadratic for E in Eq. (4.12). Doing so yields a critical field strength parameter

$$E_c=\frac{-P}{\varepsilon_a\varepsilon_0\sin^2\theta}. \quad (4.13)$$

The ranges of E, for the various signs that the parameters E, P, and ε_a can take, for which linear stability is guaranteed are shown in Tables I and II [obtained by using Eq. (4.12)]. The method therefore yields sufficient conditions on the strength of the applied static field for linear stability to hold. It should be noted that, roughly speaking, if one considers only the balance of the ferroelectric and dielectric torques in Eq. (2.20) then for $\phi\approx\pi/2$ we obtain the result in Eq. (4.13).

TABLE II. Ranges of stability for $E<0$ ($E<0$ corresponds to reversing the field) in the infinite case where E_c is given by Eq. (4.13); ε_a and P are as in Table I.

	P	
ε_a	$+$	$-$
$+$	$E_c<E<0$	method fails
$-$	$E<0$	$E<E_c$

Case (ii): Finite domain

On the finite interval D Eq. (4.9) has the solution given by Eqs. (3.59) and (3.60) (with x replaced by Z) and it therefore follows that

$$w(Z,T)=\exp([A+B]T)\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi Z}{\bar{d}}\right)\times\exp\left(\frac{-(n\pi)^2T}{\bar{d}^2}\right), \quad (4.14)$$

where $D=[0,\bar{d}]$ and $\bar{d}$ is the rescaled depth of the sample, that is,

$$\bar{d}=\frac{d}{\sqrt{B_3}} \quad (4.15)$$

with d being the original sample depth [see Eq. (2.24)]. Therefore $\pi/2$ is asymptotically exponentially stable if

$$\exp\left[\left(A+B-\frac{(n\pi)^2}{\bar{d}^2}\right)T\right] \quad (4.16)$$

decays with time. Hence we must consider the sign of

$$A+B-\left(\frac{n\pi}{\bar{d}}\right)^2. \quad (4.17)$$

Thus, for the finite case, we have linear stability provided

$$A+B-\left(\frac{n\pi}{\bar{d}}\right)^2<0,\quad n=1,2,3,\ldots. \quad (4.18)$$

Notice that, unlike the infinite case, it is possible in the finite case to have linear stability for certain positive values of $(A+B)$. It follows from (4.18) that if

$$A+B-\left(\frac{\pi}{\bar{d}}\right)^2=E^2\varepsilon_a\varepsilon_0\sin^2\theta+EP-\left(\frac{\pi}{\bar{d}}\right)^2<0 \quad (4.19)$$

then inequality (4.18) will necessarily be satisfied and linear stability will be guaranteed. Thus, on solving (4.19) in terms of E, we obtain the critical field strengths

$$E_{\pm}^{c}=\frac{-P\pm P\sqrt{1-4\varepsilon_a\varepsilon_0\sin^2\theta\pi^2/(P\bar{d})^2}}{2\varepsilon_a\varepsilon_0\sin^2\theta}, \quad (4.20)$$

where E_-^c is defined to be the critical field strength that is less than E_+^c; notice that E_+^c and E_-^c will change signs depending upon the original signs of ε_a and P. The regions of E for which we have linear stability are given in Tables III and IV. We note that, on taking the limit $\bar{d}\to\infty$ in the finite critical field strength parameters, $E_{\pm}^c\to E_c$: we obtain the infinite critical field strength parameter (4.13). It should also

TABLE III. Regions of $E>0$ for which we have stability for finite samples where $E_{\pm}^{c}$ are given by Eq. (4.20), ε_a is the dielectric anisotropy of the liquid crystal, and P is the spontaneous polarization.

	P	
ε_a	+	−
+	method fails	$E_-<E<E_+$
−	$E>E_+$	$E>E_+$

be noted that the decay rate given in (4.16) is depth dependent; the larger the depth, the longer the time required for the director to relax back from the perturbed state to the equilibrium state.

B. Nonlinear stability

We now consider the stability of the $\pi/2$ solution of the fully nonlinear equation (4.4). If, as before, we restrict the initial profile of the perturbation $w_0(z)$ to be non-negative and to lie in the function space H^1, we are able by means of the relevant comparison principle to bound $w(z,t)$ below by zero. In order to apply the methods introduced in the cubic case in Secs. III B and III C to show that the zero state was nonlinearly stable to the $\pi/2$ state in Eq. (4.1), we are required to obtain a supersolution on our perturbation on some interval for T, for example $[0,T']$, satisfying

$$\overline{w}_T=\overline{w}_{ZZ}-k\overline{w},$$

$$\overline{w}(Z,0)=Cw_0(Z), \tag{4.21}$$

where k is a positive constant determined from the parameters of the problem and C is a positive constant to be chosen later. On obtaining a supersolution satisfying Eq. (4.21), it follows from the definition of a supersolution given in Eq. (3.2) that we then have the differential inequality

$$\overline{w}_T=\overline{w}_{ZZ}-k\overline{w}\geqslant\overline{w}_{ZZ}+f(\overline{w}), \tag{4.22}$$

which enables us to apply the comparison techniques introduced above and therefore show that the perturbation decays in time and thus the $\pi/2$ equilibrium state to Eq. (4.1) is stable. To obtain a bound of the form of Eq. (4.22) we must first bound $f(w)$ so that

$$f(w)\leqslant -kw, \tag{4.23}$$

for a suitable constant $k>0$.

We now consider the restrictions on the parameters A and B given in Eq. (2.22) so that inequality (4.23) and hence

TABLE IV. Regions of $E<0$ ($E<0$ corresponds to reversing the field) for which we have stability for finite samples where $E_{\pm}^{c}$ are given by Eq. (4.20); ε_a and P are as in Table III.

	P	
ε_a	+	−
+	$E_-<E<E_+$	method fails
−	$E<E_-$	$E<E_-$

(4.22) hold. We must obtain an interval $I=[0,w_{\max}]$ within which the maximum magnitude of the perturbation must lie, such that (4.23) holds. Following the nonlinear stability analysis in the cubic case, we now let $w_c(z,t)$ be the solution to Eqs. (4.4) and (4.5) such that

$$w_c(z,0)=cw_0(z), \tag{4.24}$$

where c is some positive constant. The nonlinearity to be considered here is

$$f(w)=A\sin(w)\cos(w)+B\sin(w). \tag{4.25}$$

Thus if $w_{\max}$ exists it is possible to choose c small enough so that for some number w^*

$$cw_0(z)\leqslant w^*<w_{\max}; \tag{4.26}$$

then we can apply the arguments introduced in Secs. III B and III C to show data collapse. However, if there does not exist an interval I such that $f(w)$ is negative then we are unable to apply the above argument and cannot draw any conclusions about the stability of the $\pi/2$ state.

We now consider the behavior of $f(w)$. First we note that $f(0)=0$. Thus an immediate restriction on $f(\phi)$ is that its derivative, at zero, must decrease, that is, we require that

$$f'(0)=A+B<0. \tag{4.27}$$

This restriction is exactly that obtained in the linear stability analysis above for the infinite case. The critical field strengths (and regions of stability) are therefore identical to those obtained in Eq. (4.13) (see Tables I and II). We see, from Eq. (4.25), that $f(w)$ is equal to zero if and only if

$$w=\begin{cases}0\\ \arccos\left(-\dfrac{B}{A}\right), & \left|\dfrac{B}{A}\right|\leqslant 1\\ \pi.\end{cases} \tag{4.28}$$

Thus if $|B/A|\geqslant 1$, $f(w)$ has only two roots, namely, 0 and π. However if $|B/A|<1$, $f(w)$ has three roots, as displayed in Eq. (4.28). Hence, if A and B satisfy Eq. (4.27), we are guaranteed the existence of an interval $I=[0,w_{\max})$ such that (4.23) holds. Thus for a perturbation with a given value of $w_{\max}$ the maximum value $k_{\max}$ of k can be found by solving the equality part in (4.23), which leads to

$$k_{\max}=\frac{-f(w_{\max})}{w_{\max}}. \tag{4.29}$$

However, Eq. (4.29) is a transcendental equation and cannot be solved analytically. We are nonetheless guaranteed a solution to (4.29) if we choose $w_{\max}$, small enough. The right hand side of Eq. (4.29) is an even function in $w_{\max}$ and we therefore need only consider positive values of $w_{\max}$. Hence there must exist a constant $k\leqslant k_{\max}$, which depends upon the constants A, B, and $w_{\max}$, such that the solution to Eq. (4.21) is a supersolution. From the qualitative features of Fig. 4, $w_{\max}$ can be lowered or raised, depending on the values of A and B.

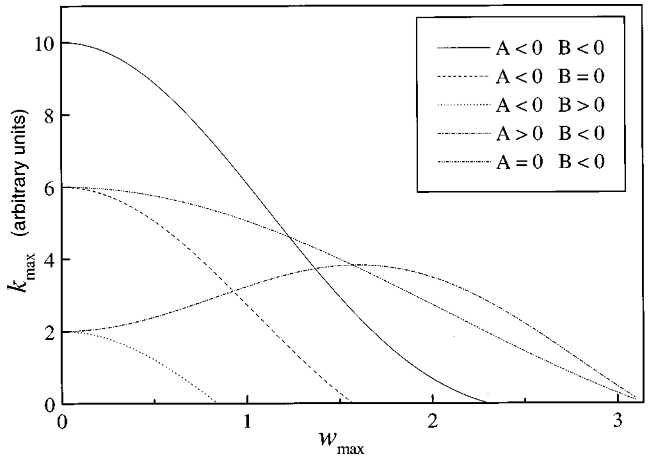

FIG. 4. Qualitative plots of Eq. (4.29) showing k_{max} for a given maximum magnitude w_{max} of the perturbation $w(z,t)$ for the signs of A and B displayed above.

Case (i): Infinite domain

As in the cubic case on the infinite domain, the solution to Eq. (4.21) is given by

$$\overline{w}(Z,T)=\frac{e^{-kT}}{\sqrt{4\pi T}}\int_{-\infty}^{\infty}\exp\left(\frac{-(Z-y)^2}{4T}\right)Cw_0(y)dy, \tag{4.30}$$

which satisfies

$$\overline{w}(Z,0)=Cw_0(Z)\geqslant cw_0(Z),$$

provided $C\geqslant c$. It therefore follows, by applying the comparison principle for infinite domains, that

$$w_c(Z,T)\leqslant\overline{w}, \tag{4.31}$$

where $\overline{w}$ is given in Eq. (4.30). Having obtained this bound it now follows, exactly as in the cubic case in Sec. III B, that in the infinite case the L_2 norm and thus the solution itself decrease in time; therefore the $\pi/2$ state is nonlinearly stable for initial perturbations in L_2 satisfying the inequality (4.23).

Case (ii): Finite domain

For the finite interval case $D=[0,\overline{d}]$, where we have rescaled using Eqs. (2.24) and (2.23), the solution to Eq. (4.4) is given by

$$\overline{w}(Z,T)=\exp(-kT)\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi Z}{\overline{d}}\right)\exp\left[-\left(\frac{n\pi}{\overline{d}}\right)^2T\right], \tag{4.32}$$

where the A_n are as given in Eq. (3.60) with w_0 playing the role of u_0. Assuming that $w_c(Z,T)$ vanishes at $z=0$, $\overline{d}$ for $t\geqslant 0$, we also have that

$$\overline{w}(Z,0)=Cw_0(Z)\geqslant cw_0(Z)=w_c(Z,0). \tag{4.33}$$

Hence it follows, from the comparison principle on a finite domain, that

$$w_c(T)\leqslant\exp(-kT)\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi Z}{\overline{d}}\right)\exp\left[-\left(\frac{n\pi}{\overline{d}}\right)^2T\right], \tag{4.34}$$

and thus, using a similar argument to that used previously in the finite cubic case in Sec. III C, we can uniformly bound each of the A_n given by Eq. (3.60). Thus for stability we require

$$k+\left(\frac{n\pi}{\overline{d}}\right)^2>0, \tag{4.35}$$

which, since k is positive, is always the case. It therefore follows that if A and B satisfy Eq. (4.27) then solutions corresponding to small enough initial data collapse to the $\pi/2$ state. Note here that, similar to the linear analysis, the restriction (4.35) depends upon the rescaled depth $\overline{d}$ of the sample. It follows that the bounds on the relaxation time are depth dependent, as can be seen from (4.34).

V. DISCUSSION

Restrictions for the decay of a non-negative perturbation $w(z,t)\in H^1$ initially applied to a constant equilibrium solution of the dynamic equation (2.21) (on both infinite and finite domains) were considered. This dynamic equation governs the director reorientation within a sample of ferroelectric smectic-C liquid crystal, where the static field is applied parallel to the smectic planes. Theoretical critical field strengths (related to the stability of perturbations), which depend upon the physical parameters of the problem, were then obtained. Having obtained these critical field strengths we then, for certain values of these parameters, obtained ranges of the applied static field for which, on both finite and infinite domains, linear and nonlinear stability are guaranteed. Upper bound estimates upon the relaxation time of the director were also obtained for given initial maximum magnitudes of the perturbation.

In Sec. III B we employed the comparison techniques introduced in Sec. III A to establish, as considered by Flores [4], the stability of certain solutions to a reaction-diffusion equation involving a cubic nonlinearity on an infinite domain. This stability argument was then adapted in Sec. III C for application to solutions on a finite interval.

Finally, in Sec. IV a perturbation method was introduced to consider the stability of the $\pi/2$ equilibrium state of the dynamic equation (2.21), which governs the orientation of the director when the static electric field is applied parallel to the layers. The techniques used to prove stability for the cubic reaction-diffusion equation in Secs. III B and III C were then applied to the linearized and fully nonlinear dynamic perturbation equation. However, unlike for the cubic cases discussed in Sec. III, qualitative information was obtained on the parameters of the liquid crystal for which stability holds; critical field strengths and ranges of the static electric field within which linear and nonlinear stability hold were found. For the ranges of the parameters for which stability holds, upper bound estimates were obtained on the characteristic time taken for the director to relax back to its unperturbed state. It is possible to obtain information on the

TABLE V. Characteristic times τ, which are all positive, for the problems discussed in Sec. IV [compare with Eqs. (4.11), (4.16), (4.30), and (4.35)]. λ_5 is the positive viscosity coefficient discussed in the text, B_3 is a positive elastic constant, and d is the original sample depth; the field dependent contributions are provided by $A+B=E^2\varepsilon_a\varepsilon_0\sin^2(\theta)+PE$ and k, as introduced in Sec. IV, subject to E satisfying the stability conditions in Tables I–IV.

Linear		Nonlinear	
Finite domain	Infinite domain	Finite domain	Infinite domain
$\tau_n=\dfrac{2\lambda_5 d^2}{B_3(n\pi)^2-(A+B)d^2}$	$\tau_\infty=-\dfrac{2\lambda_5}{(A+B)}$	$\tau_n=\dfrac{2\lambda_5 d^2}{kd^2+B_3(n\pi)^2}$	$\tau_\infty=\dfrac{2\lambda_5}{k}$

usual characteristic time τ for the various problems considered. From Eqs. (4.11) and (2.23) we find that τ for the relaxation of the director in the infinite linear case is given by $\tau_\infty=-2\lambda_5/(A+B)=-2\lambda_5/[E^2\varepsilon_a\varepsilon_0\sin^2(\theta)+PE]$ (notice that $A+B$ is necessarily negative in the infinite domain case: see Secs. III and IV for more details). Similar results, which are displayed in Table V where d is the original sample depth, can also be found by considering Eqs. (4.16), (4.30), and (4.35). The subscript n in the finite cases indicates the value of τ relating to the nth mode in the corresponding series solution. The first eigenmode relates to the longest characteristic time (this is easily seen by letting n become large in either of the two finite case characteristic times in Table V). It therefore follows that τ_1 is the most influential.

Table V gives an indication of how long it takes for the director to relax back to the equilibrium $\phi=\pi/2$ of Eq. (4.1): the larger the value of τ, the longer the time taken for the director to equilibrate. Since, in the finite cases, it is the first eigenmode that yields the largest characteristic time, it follows that τ_1 is indicative of the time taken for the director to relax.

The characteristic times for the linear and the nonlinear analysis are analogous to each other: although there is no depth dependence in the infinite cases (as is to be expected) the sample depth and the eigenvalues play a role in the finite cases. In both the linear and nonlinear analysis τ_∞ may be obtained by taking the limit $d\to\infty$ in each of the corresponding finite case characteristic times. A simple calculation reveals that in both cases (finite and infinite) τ_1 increases monotonically to the corresponding τ_∞ as the sample depth d (or $\bar{d}$) is increased. Thus the time taken for the director to relax back to its equilibrium state from its perturbed state is increased as the sample depth d (or $\bar{d}$) increases. Similarly, on taking the limit close to $d\approx 0$ in each of the finite case characteristic times, we find that as the sample depth is decreased τ approaches zero. Hence to minimize the time taken for a perturbed sample to return to its unperturbed state the original sample depth d should be made as small as possible.

It is not, however, only the sample depth that plays a role in the magnitude of the characteristic times. In both the linear and nonlinear analysis it can be seen that the magnitudes of electric field dependent terms $(A+B)$ and k [which is a function of $(A+B)$] also influence the characteristic time. If the electric field is close to zero or E_c, given by Eq. (4.13), we see from Eq. (4.12) that if $(A+B)$ is small then the characteristic time becomes large. Note also that in the infinite case the elastic constant B_3 does not appear. This is not unexpected as this elastic constant is absorbed, via rescaling, to the spatial variable Z and thus it cannot enter the characteristic times, as there is no boundary in the infinite case. This is certainly the case in other problems involving infinite domains where solutions are considered in smectic-C or smectic-C^* liquid crystals, where it is known from exact traveling wave solutions that the wave speed is independent of the elastic constants [16,20].

There are only a few known results for characteristic times for ferroelectric smectic-C samples arranged as discussed in the above problem. For example, Abdulhalim, Moddel, and Clark [[26], p. 823] discuss a characteristic elastic time,

$$\tau_e=\frac{\eta_\phi}{q^2K_s}, \tag{5.1}$$

where η_ϕ is a typical smectic viscosity, K_s is a smectic constant, and q is a typical wave number: in finite domains q can simply be considered as a ‘‘first’’ wave number $q=\pi/L$ where L is the sample depth. Numerical results are also given in [26]. In the finite domain cases in Table V, τ_1 is of a similar form except for the $(A+B)d^2$ and kd^2 contributions: these additional terms arise from the physical parameters of the sample being smectic-C rather than smectic-A. The results presented here are therefore consistent with those anticipated by Abdulhalim *et al.* [26].

Also, when the smectic tilt angle $\theta=0$ (see Fig. 1) the sample becomes smectic-A type, for which there are recent results by Shalaginov, Hazelwood, and Sluckin [27] for various types of relaxation phenomena. These results, although for smectic-A, can be compared with both the finite domain linear and nonlinear cases outlined in Table V. From [27], there is a typical characteristic time τ_v given by a similar form to (5.1), namely,

$$\tau_v=\frac{\eta_3L^2}{4\pi^2K}, \tag{5.2}$$

L being the sample depth, η_3 a viscosity, and K an elastic constant: a typical value for τ_v is around 10^{-2} s [27]. (In liquid crystals characteristic times are frequently proportional to the ratio of a viscosity divided by an elastic constant [9,28].) Clearly, the expression in Eq. (5.2) bears some resemblance to the results for τ_n in Table V, where d, B_3, and $2\lambda_5$ play the roles of L, K, and η_3. The factor of 4 appearing in the denominator of τ_v occurs because the authors in [27] consider the second mode in their analysis when looking for the initialization of chevrons. When such terms as $(A+B)d^2$ and kd^2 are ignored, for example, when the smectic-*C* sample is close to smectic-*A* (i.e., $\theta \approx 0$) then the results in [26,27] can also be utilized for a comparison with the results presented here, bearing in mind that these authors employ the analog of τ_2. The characteristic times for τ_1 in Table V for ferroelectric smectic-*C* samples ought to collapse to those for smectic-*A* (with slightly different notation) when the smectic-*C* contributions are neglected. These results for ferroelectric smectic-*C* liquid crystals are therefore expected to be natural extensions to results for smectic-*A* in special cases, the characteristic times being modified according to the forms indicated in Table V.

It should also be possible to apply the methods used in Sec. IV to other dynamic equations that appear in liquid crystal theory. In particular, it may be possible to obtain information on the stability of some of the equilibrium states that arise when a sample of smectic-*C* or smectic-*C** liquid crystal has a static electric field applied at an angle to the smectic layers.

Equations similar to those discussed here occur elsewhere [16,20] and other additional sinusoidal terms may be included in the governing equation, similar to those that arise (in a different context) in the results contained in [26]. It should also be mentioned that Stewart and Faulkner [29] have obtained stability results for nonconstant traveling waves in nematic liquid crystals on infinite domains arising from a cubic equation similar to Eq. (3.8). Work on this and related areas requires a different analysis and is currently in progress by the authors.

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